§ 15.8

$$\iint_R f(x,y)\, \begin{matrix} dx\,dy \\ dy\,dx \end{matrix} = \iint_G f(u,v)\, |J(u,v)| \begin{matrix} du\,dv \\ \text{or} \\ dv\,du \end{matrix}$$

Jacobian

$\boxed{\begin{matrix} x(u,v) \\ y(u,v) \end{matrix}}$ transformation

G is the image of R

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

ex | Find the Jacobian for the transformation from cartesian to polar.

$x = r\cos\theta$

$y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

example 2

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_0^2 \int_0^1 2u\,du\,dv = \cdots = 2$$

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2} \implies y = 2v, \quad x = u+v$$

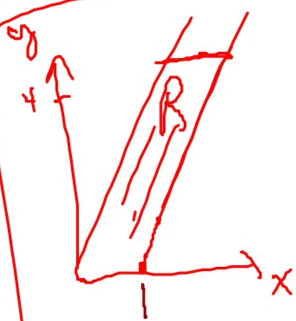
$0 \le y \le 4$

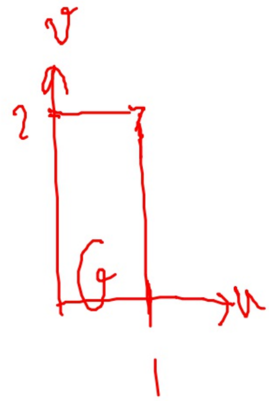
$0 \le v \le 2$

$\frac{y}{2} \le x \le \frac{y}{2}+1$

$v \le u+v \le v+1 \Rightarrow 0 \le u \le 1$

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$





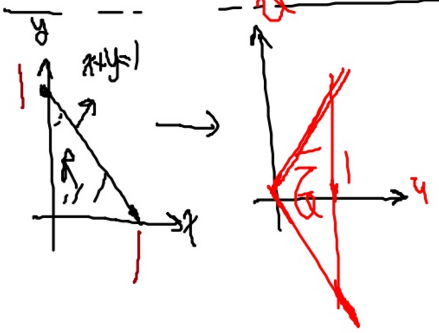
example 3

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2 \left(\frac{1}{3}\right) dv\,du$$

$$\left.\begin{array}{l} u = x+y \\ v = y-2x \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{array}$$

$$J = \begin{vmatrix} 1/3 & -1/3 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$= \ldots$



$x+y=1 \Rightarrow u=1$

$x=0 \Rightarrow u=v$

$y=0 \Rightarrow v=-2u$

ex 6. Find the Jacobian of transformation from Cart. to Cyl.

$x = r\cos\theta$; $y = r\sin\theta$; $z = z$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

ex c   Cart. to spherical: $z = \rho\cos\phi$; $x = \rho\sin\phi\cos\theta$; $y = \rho\sin\phi\sin\theta$.

$$J = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} =$$

$$\cos\phi\left(\underline{\rho^2\sin\phi\cos\phi}\cos^2\theta + \underline{\rho^2\sin\phi\cos\phi}\sin^2\theta\right) =$$

$$+\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$\rho^2\sin\phi\cos^2\phi + \rho^2\sin\phi\sin^2\phi = \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) = \rho^2\sin\phi \geq 0$$

$$0 \leq \phi \leq \pi$$

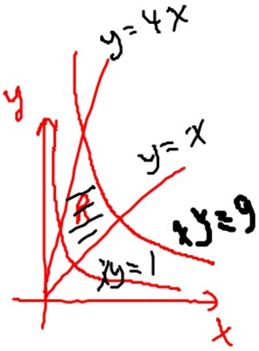
9) $\displaystyle\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^2 \int_1^3 (u+v)\,\frac{2u}{v}\; du\,dv$

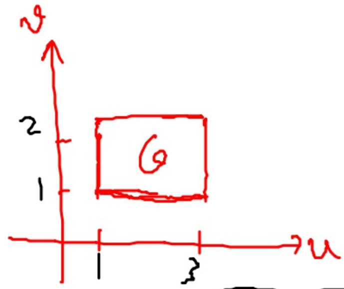
$= \cdots = 8 + \frac{52}{3}\ln 2$

R: $xy = 1$, $xy = 9$, $y = x$, $y = 4x$

$\frac{y}{x} = 1 \Leftarrow y = x$

$\frac{y}{x} = 4 \Leftarrow y = 4x$





$x = \frac{u}{v}$

$y = uv$

$u > 0$
$v > 0$

$\frac{y}{x} = v^2$

$xy = u^2$

$xy = 1 \Rightarrow u = 1$

$xy = 9 \Rightarrow u = 3$

$y = x \Rightarrow v = 1$

$y = 4x \Rightarrow v = 2$

$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v} \geq 0$

$\boxed{|J| = \frac{2u}{v}}$

14. $\int_0^2 \int_{y/2}^{\frac{y}{2}+2} y^3 (2x-y)\, e^{(2x-y)^2}\, dx\, dy = \int_0^2 \int_0^2 v^3 (2u)\, e^{4u^2} (1)\, du\, dv$

$$\boxed{\begin{aligned} x &= u + \frac{v}{2} \\ y &= v \\ 2x - y &= 2u \end{aligned}}$$

$$J = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$$

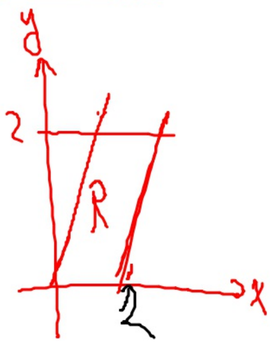
$$= \left(\int_0^2 v^3\, dv\right)\left(\int_0^2 2u\, e^{4u^2}\, du\right)$$

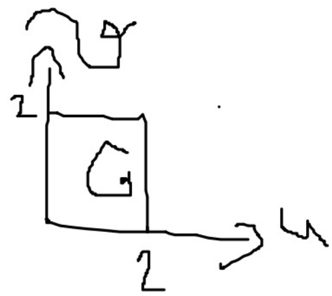
$$= e^{16} - 1 \quad ??$$

$x = \frac{y}{2} \Rightarrow u = 0$?

$x = \frac{y}{2} + 2 \Rightarrow u = 2$?

$0 \le y \le 2 \Rightarrow 0 \le v \le 2$





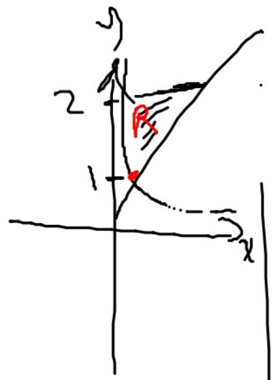
example 4 : $\displaystyle\int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\,dy = \dots$

$u = \sqrt{xy}$

$v = \sqrt{y/x}$

$uv = \sqrt{y^2} = y$

$\frac{u}{v} = \sqrt{x^2} = x$



$y = x \Rightarrow \frac{y}{x} = 1$
$\Rightarrow v = 1$

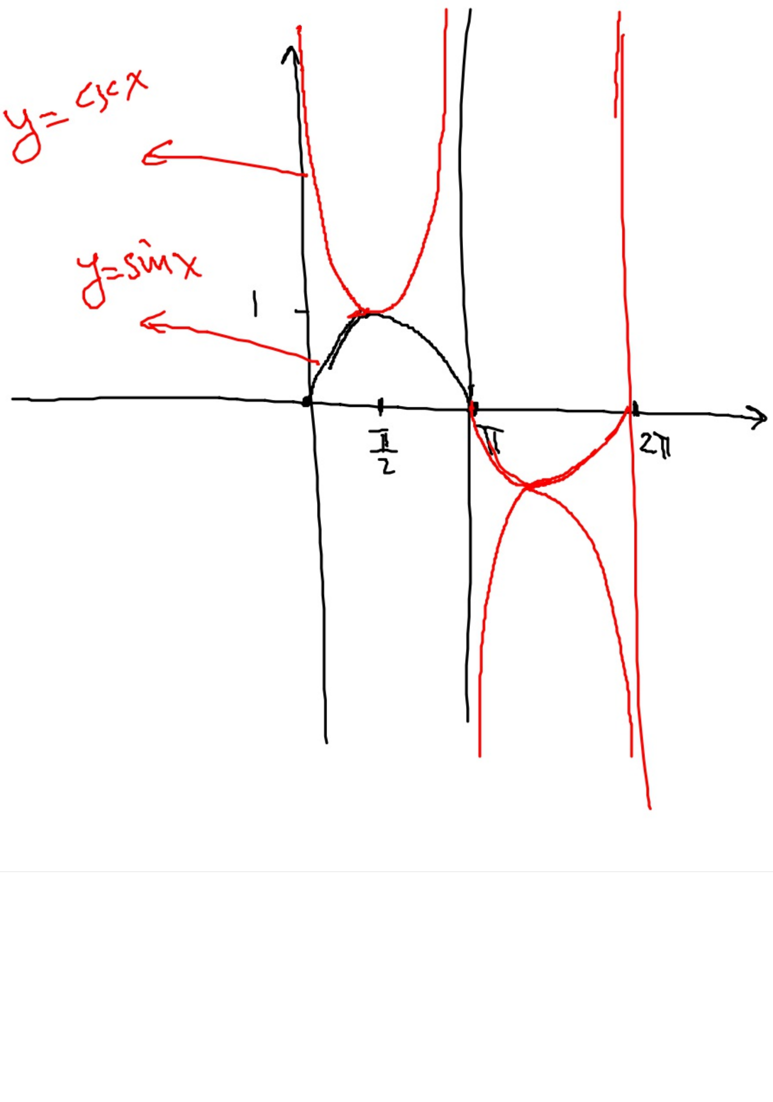
$xy = 1 \Rightarrow u = 1$

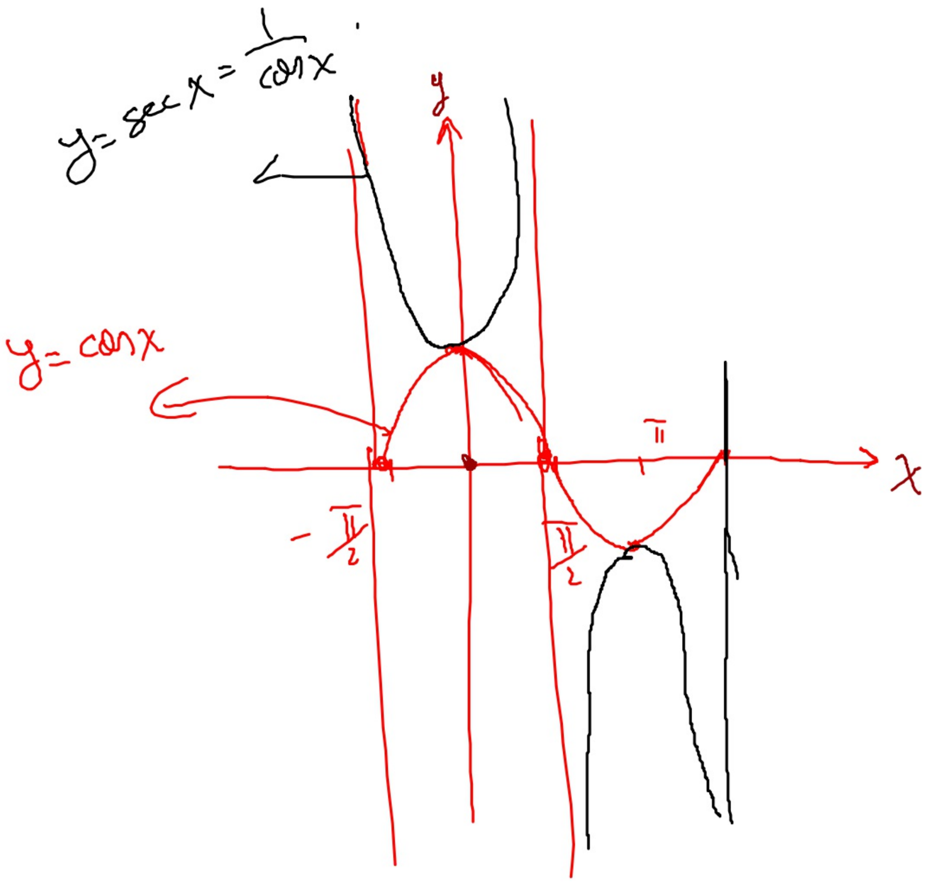
$y = 2 \Rightarrow uv = 2$

$$= \int_1^2 \int_1^{\frac{2}{u}} \cancel{v} e^{u}\, \frac{2u}{\cancel{v}}\, dv\,du$$

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v} > 0$$

Graphing : $y = \sin x$ & $y = \frac{1}{\sin x} = \csc x$.





y = sec x = 1/cos x
y = cos x
y
x
-π/2
π/2
π